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Bäcklund Transformations, Solitary Waves, Conoid Waves and Bessel Waves of the (2 + 1)-Dimensional Euler equation

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Some simple special Bäcklund transformation theorems are proposed and utilized to obtain exact solutions for the (2 + 1)-dimensional Euler equation. It is found that the (2 + 1)-dimensional Euler equation possesses abundant soliton or solitary wave structures, conoid periodic wave structures and the quasi-periodic Bessel wave structures on account of the arbitrary functions in its solutions. Moreover, all solutions of the arbitrary two dimensional nonlinear Poisson equation can be used to construct exact solutions of the (2 + 1)-dimensional Euler equation.

KEY WORDS: Bäcklund transformations theorems; (2 + 1)-dimensional Euler equation; Solitary waves; Conoid periodic wave; Bessel wave.

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1. INTRODUCTION

It is well known that the most general governing equation in fluid dynamics is the Navier-Stockes (NS) equation (Fefferman, 2000; Groisman and Quake, 2004; Pedrizzetti, 2005; Saveliev and Gorokhovski, 2005; Sundkvist *et al.*, 2005). In many cases, the viscocity of the fluids is quite small and may be neglectable, which reduces the NS equation to the Euler equation. In addition to its fundamental

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application in fluids, the Euler equation can also manipulate many other physical fields such as the plasmas, condense matters, astrophysics, etc. (Babaev *et al.*, 2004; Bergmans and Schep, 2001; Bonazzola *et al.*, 1997; Cafaro *et al.*, 1998; Canuto and Dubovikov, 2005; Chavanis, 2000; Chavanis and Sommeria, 1997; Chuang *et al.*, 1991; Del Sarto *et al.*, 2003; Faddeev *et al.*, 2003; Girard *et al.*, 2005; Gluhovsky and Agee, 1997; Grasso *et al.*, 2001; Haldane and Wu, 1985; Huang *et al.*, 1998; Kurien *et al.*, 2000; Kuvshinov *et al.*, 1994, 1999; Leggett, 2001; Marshall *et al.*, 1997; Niemi, 2005; Thiffeault and Morrison, 2002; Weichman and Petrich, 2001; Xu *et al.*, 2005). Consequently, to find exact solutions of the Euler equation is very important and crucial both in mathematics and in real physical applications. This equation is known to have a lot of exact analytical solutions. Some of them can be found in the classical book of H. Lamb (1945). In Abrashkin and Yakubovich (1984), the authors have studied the planar rotational flows of an ideal fluid and the addressing method is developed to get exact solutions of the Euler equation in Yurov and Yurova (2006).

Under some different approximate assumptions, various integrable models such as the Burgers, KdV, modified KdV, nonlinear Schrödinger, KP and DS equations can be derived form the Euler or NS equation (Calogero and Ji, 1991, 1993; Kivshar and Malomed, 1989). Beginning with these integrable models, the existences of different kinds of solitons and/or solitary waves are proved and/or predicted.

This paper aims to find some types of exact solutions of the Euler equation in an alternative way. The results are derived by a simple and essential technique, Bäcklund transformation. The Bäcklund transformation approach has been widely used to find exact solutions for various integrable systems. There are some excellent books on the Bäcklund transformations, say (Rogers and Schief, 2002).

In Li (2001, 2003); Li and Shvidkoy (2004); Li and Yurov (2003), the authors write the (2 + 1)-dimensional Euler equation in a vorticity form

$$\Omega = \psi_{xx} + \psi_{yy} \equiv \triangle\psi, \tag{1}$$

$$\Omega_t + [\psi,\ \Omega] = 0, \quad [\psi,\ \Omega] \equiv \psi_x\Omega_y - \psi_y\Omega_x, \tag{2}$$

where Ω is the vorticity, ψ is the stream function and they are linked to the velocity field $\{u,\ v\}$ by

$$u = -\psi_y, \quad v = \psi_x, \quad \Omega = v_x - u_y = \triangle\psi. \tag{3}$$

We will write down the Bäcklund transformation theorems for the Euler equation (1)–(2) in the next section. With the help of the Bäcklund transformations, some types of exact solutions, including the travelling wave solutions, solitary waves, different types of the conoid periodic waves, the Bessel waves and the interaction solutions of the conoid periodic waves with some related explicit figures are also delivered in Section 2. Lastly, the paper ends with brief summary and discussions.

2. BÄCKLUND TRANSFORMATIONS AND SPECIAL EXACT SOLUTIONS

Theorem 1. *If* $\{\Omega_0,\ \psi_0\}$ *is a solution of the Euler equation (1)–(2), so is* $\{\Omega_1,\ \psi_1\}$ *under the definition*

$$\Omega_1 = \Omega_0 + q, \quad \psi_1 = \psi_0 + p, \tag{4}$$

where $\{p,\ q\}$ *is a solution of*

$$q = \triangle p, \tag{5}$$

$$q_t + [p,\ q] + [\psi_0,\ q] + [p,\ \Omega_0] = 0. \tag{6}$$

Proof: Direct calculations.

The next key step to find exact solutions from known ones is to make some further ansatzs for the functions q and/or p. In the present stage, we put a constraint between q and p as general as

$$q = Q(p). \tag{7}$$

Using the above assumption (7) and Theorem 1, we may have a much simplified Bäcklund transformation theorem:

Theorem 2. *If* $\{\Omega_0,\ \psi_0\}$ *is a solution of the Euler equation (1)–(2), so is* $\{\Omega_1,\ \psi_1\}$ *with the definition*

$$\Omega_1 = \Omega_0 + Q(p), \quad \psi_1 = \psi_0 + p, \tag{8}$$

where $Q(p)$ *is an arbitrary function and* p *is a solution of the over-determined equation system*

$$\triangle p = Q(p), \tag{9}$$

$$p_t + [\psi_0,\ p] = 0, \tag{10}$$

$$[p,\ \Omega_0] = 0. \tag{11}$$

Proof: Substituting the assumption (7) to the equation (6) of the theorem 1, we have

$$(p_t + [\psi_0,\ p])Q_p + [p,\ \Omega_0] = 0, \tag{12}$$

where $Q_p \equiv \frac{\mathrm{d}Q}{\mathrm{d}p}$. It is straightforward to see that (12) is correct for *arbitrary* $Q(p)$ iff (if and only if) (10) and (11) are satisfied simultaneously. As for Eq. (9), it is just a simple combination of (5) and (7). Therefore, the Theorem 2 is completely proved.

The equivalent result of the Theorem 2 can also be obtained by other methods, say the dressing method (Yurov and Yurova, 2006).

Although the Bäcklund transformation Theorem 2 has been much simplified, it is still rather difficult to construct some exact solutions via it with a complicated known solution. So here, we just make use of a very special and simple seed solution, namely, the constant vorticity solution

$$\Omega_0 = \omega \qquad (\omega \text{ is constant}) \tag{13}$$

to get some significant solutions of the (2 + 1)-dimensional Euler equation.

It is easy to verify that (11) is identically satisfied under the constant vorticity solution (13) and that the corresponding general solution for the stream function ψ_0 has the form of ($i \equiv \sqrt{-1}$)

$$\psi_0 = \frac{\omega}{4}(x^2 + y^2) + f_1(x + iy,\ t) + f_2(x - iy,\ t), \tag{14}$$

where $f_1 \equiv f_1(x + iy,\ t) \equiv f_1(z, t)$ and $f_2 \equiv f_2(x - iy,\ t) \equiv f_2(z^*, t)$ are arbitrary functions of the indicated variables. Obviously, ψ_0 is assured to be real on condition that f_2 is a complex conjugate of f_1 and vice versa.

Substituting (14) into (10), we have

$$p_t + \frac{\omega}{2}(xp_y - yp_x) + if_{2z^*}(p_x - ip_y) - if_{1z}(p_x + ip_y) = 0, \tag{15}$$

i.e.,

$$ip_t + \frac{\omega}{2}(z^* p_{z^*} - zp_z) + 2(f_{1z}p_{z^*} - f_{2z^*}p_z) = 0. \tag{16}$$

In order to solve the remaining equation (15) or (16) and then (9), we need to specify $f_1,\ f_2$ and ω further.

2.1. Exact Solutions Obtained from the Time Dependent Homogeneous Flow

It is clear that the $(2+1)$-dimensional Euler equation (1)–(2) permits a time dependent homogeneous flow solution

$$u = -b(t), \quad v = a(t), \tag{17}$$

i.e.,

$$\psi_0 = a(t)x + b(t)y, \quad \Omega_0 = 0, \tag{18}$$

where $a(t)$ and $b(t)$ are arbitrary functions with respect to time t. Accordingly, in the present case, the arbitrary constant and functions in the vorticity and stream function solutions (13) and (14) are determined as

$$\omega = 0, \quad f_1 = \frac{1}{2}(a(t) - ib(t))z, \quad f_2 = \frac{1}{2}(a(t) + ib(t))z^*.$$

Hence, (15) (or (16)) becomes

$$p_t + a(t)p_y - b(t)p_x = 0, \tag{19}$$

whose general solution possesses the form

$$p = p\left(x + \int b(t)\mathrm{d}t,\ y - \int a(t)\mathrm{d}t\right) \equiv P(X,\ Y), \tag{20}$$

where $P(X,\ Y)$ is an arbitrary function of $\{X,\ Y\}$ and will be determined by substituting (20) into (9).

The above result can be summarized as the following theorem:

Theorem 3. *If $P(X,\ Y) \equiv P$ is a solution of the nonlinear Poisson equation*

$$P_{XX} + P_{YY} = Q(P) \tag{21}$$

with $Q(P)$ being an arbitrary function of P and $X = x + \int b(t)\mathrm{d}t,\ Y = y - \int a(t)\mathrm{d}t$, then $\{\Omega,\ \psi\}$ given by

$$\Omega = Q(P), \tag{22}$$

$$\psi = a(t)x + b(t)y + P \tag{23}$$

is a solution of the $(2+1)$-dimensional Euler equation (1)–(2).

Generally, under any given function $Q(P)$, the nonlinear Poisson equation (21) (similar to the nonlinear Klein-Gordon equation) is nonintegrable. However, one can still find some exceptions that (21) is integrable only if $Q(P)$ is one of the following types of expressions

$$a_0 + a_1 P, \quad a_0 e^{a_1 P}, \quad a_0 \sin(a_1 P),$$

$$a_0 \sinh(a_1 P), \quad a_0 e^{-a_1 P} + a_2 e^{2a_1 P},$$

where $a_0,\ a_1$ and a_2 are arbitrary constants.

It is noted that a special case of Theorem 3 when $a(t)$ and $b(t)$ are just arbitrary constants has been deduced from a Darboux transformation with zero spectral parameter (Lou and Li, 2006).

Now we manage to obtain many soliton, solitary wave and conoid periodic wave solutions for the (2 + 1)-dimensional Euler equation (1)–(2) through different selections of $Q(P)$. Here, we just list some special examples without further details on the calculations.

Example 1 Travelling wave solutions of the Poisson equation. The general form of the travelling wave solution of the Poisson equation can be written as an integral form

$$\int^P \sqrt{\frac{(1+c^2)}{2\int Q(\mu)\mathrm{d}\mu - V_0}}\mathrm{d}\mu = X_0, \tag{24}$$

where

$$X_0 = x + cy + \int [b(t) - ca(t)]\mathrm{d}t - x_0, \tag{25}$$

and $c,\ x_0,\ V_0$ are arbitrary constants.

For some special types of selections of Q and the integral constant V_0, the integral form (24) can be rewritten as some types of Jacobi elliptic functions. For instance,

$$P = P_1 = 4\arctan\left(\sqrt{n}\ \mathrm{sn}\frac{\sqrt{m}\ X_0}{(1+n)\sqrt{1+c^2}}\right) \tag{26}$$

for the Poisson sine selection

$$Q(P) = -m \sin P, \tag{27}$$

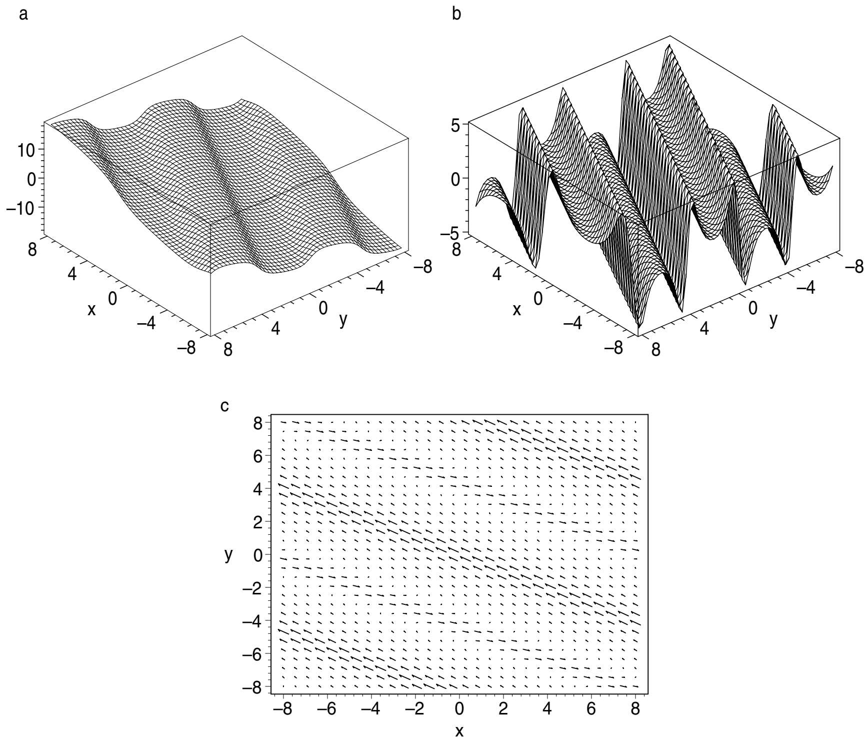

Fig. 1. (a) A special conoidal sn wave for the stream function ψ given by (23) with (25), (26) and (28) at time $t = 0$. (b) and (c) are the vorticity and velocity structures corresponding to (a).

where n is the modulus of the Jacobi sn function and the corresponding V_0 reads

$$V_0 = \frac{2m(n^2 - 6n + 1)}{n^2 + 1}.$$

Figure 1a displays the structure for the stream function ψ (23) with (26) under the function and parameter selections

$$a(t) = b(t) = m = 1, \; c = 2, \quad n = 0.9 \tag{28}$$

at time $t = 0$. The corresponding vorticity structure Ω and the velocity field $\{u = -\psi_y, \; v = \psi_x\}$ are shown in Fig. 1b and c respectively.

It is known that when the modulus n of the Jacobi elliptic function tends to 1, the conoidal wave tends to a soliton or solitary wave solution. Figure 2a–c exhibit the solitary wave structure for the stream, vorticity and velocity respectively under the same function and parameter selections (28) except for $n = 1$.

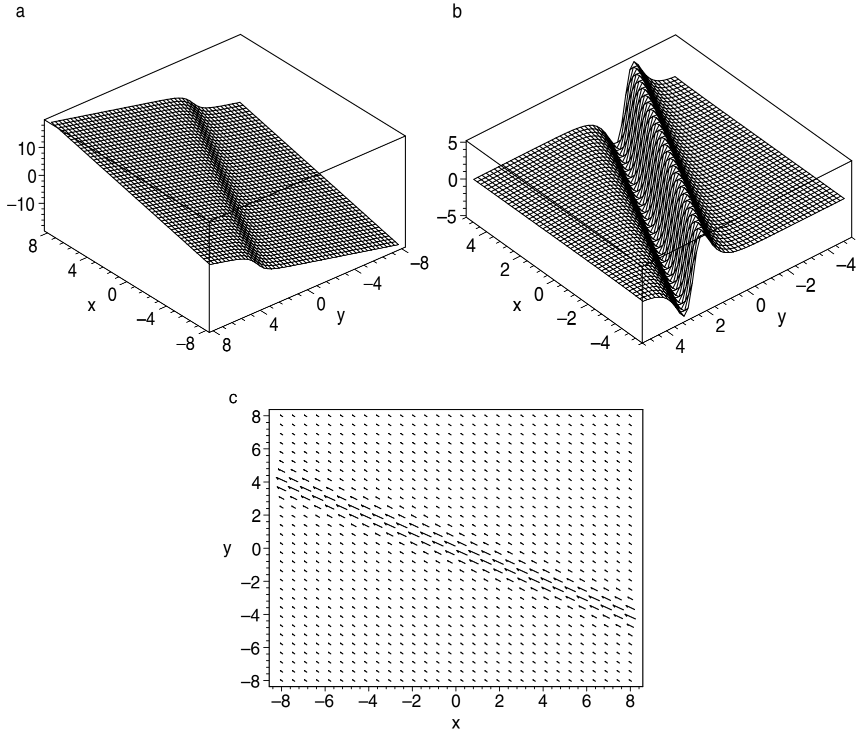

Fig. 2. A special soliton structure for (a) the stream function, (b) the voticity and (c) velocity vector which are limit cases of Fig. 1a–c respectively with $n = 1$.

Naturally, different selections of the arbitrary function $Q(P)$ will lead to different type of periodic and solitary waves (Hesthaven *et al.*, 1995).

Example 2 Interactions among two conoidal periodic waves. To find non-travelling wave solutions of the nonlinear Poisson equation (21) is much more involved and complicated.

Here we only write down one special two conoidal periodic wave interaction solution which is *nonsingular* for the Poisson sine equation. It is easy to prove that if we choose the function $Q(P)$ as the form of

$$Q(P) = \frac{c_1^2 + c_2^2}{n_1}\left(n_1\left(2 - n_2^2\right) + \left(2n_1^2 - 1\right)\sqrt{\frac{1 - n_2^2}{1 - n_1^2}}\right)\sin P \tag{29}$$

with four arbitrary constants $c_1,\ c_2,\ n_1$ and n_2, then a two-conoidal periodic wave interaction solution can be produced

$$P = 4\arctan\left(\sqrt[4]{\frac{n_1^2(1-n_2^2)}{1-n_1^2}}\ \frac{\mathrm{cn}(X_1,\ n_1)}{\mathrm{dn}(Y_1,\ n_2)}\right), \tag{30}$$

where arbitrary constants n_1 and n_2 are the moduli of the Jacobi cn and dn functions respectively and

$$X_1 = \sqrt[4]{\frac{1-n_2^2}{n_1^2(1-n_1^2)}}\left[c_1x + c_2y + \int (c_1b(t) - c_2a(t))\mathrm{d}t\right],$$

$$Y_1 = c_2x - c_1y + \int (c_2b(t) + c_1a(t))\mathrm{d}t.$$

Figure 3 depicts a special picture on the conoidal cn-dn interaction wave given by (30) with the special selections

$$a(t) = b(t) = c_1 = 1, \quad c_2 = 1.5, \quad n_1 = n_2 = 0.99 \tag{31}$$

at time $t = 0$, with the structures of the stream function, voticity and velocity vector field distributed respectively in Fig. 3a–c.

In fact, for the $(2+1)$-dimensional sine-Gordon equation, many kinds of *singular* double periodic solutions have been found in literature, say, (Vitanov, 1996).

If we select $Q(P)$ as the sinh function instead of the sine function, some authors have obtained many double periodic solutions (Chow *et al.*, 2003; Gurarie and Chow, 2004; Kuvshinov and Schep, 2000). So we will not list the exact solutions on the Poison sinh equation here though the independent variables are different in this paper.

2.2. Exact Special Solutions Obtained from the Constant Nonzero Vorticity Field

In principle, the stream function corresponding to a constant nonzero vorticity is in the form (14). In order to get some explicit results, we choose the arbitrary functions f_1 and f_2 as simple as

$$f_1 = (a + ib)z, \qquad f_2 = f_1^* = (a - ib)z^*, \tag{32}$$

where a and b are arbitrary constants.

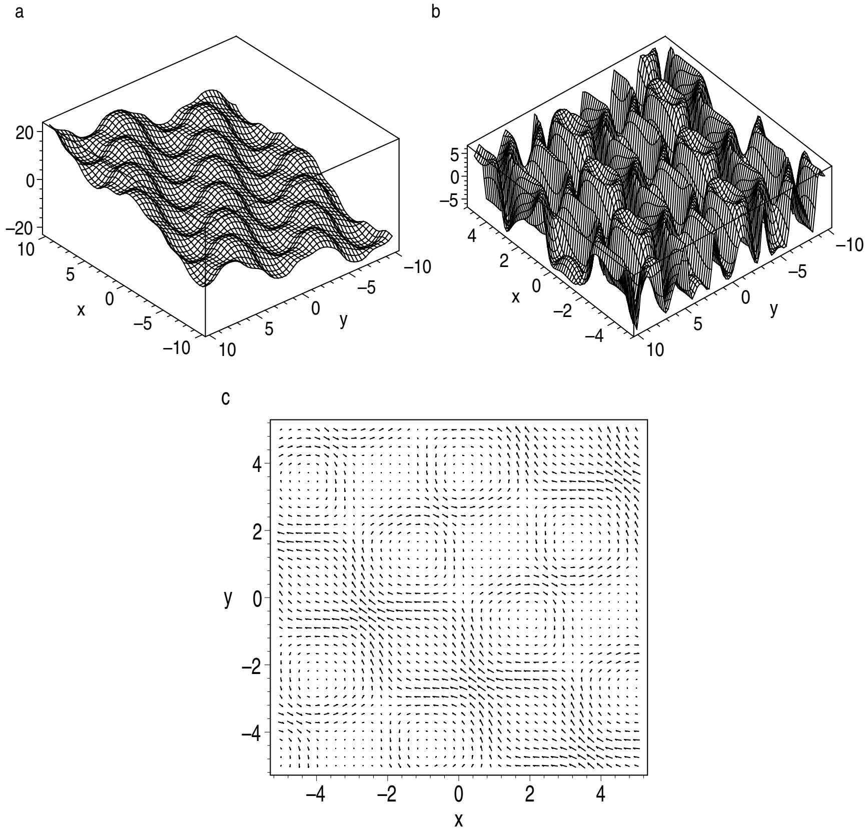

Fig. 3. (a) The conoidal cn-dn wave interaction solution for the stream function of the $(2+1)$-dimensional Euler equation described by (23) with (30) and (31) at time $t=0$. (b) The corresponding vorticity structure related to (a). (c) The velocity vector field related to the stream shown in (a).

Substituting (32) into (15), we can solve out that

$$
\begin{aligned}
&p = P(\xi,\ \eta),\\
&\xi \equiv \frac{1}{\omega}[(\omega x + a)^2 + (\omega y - b)^2],\\
&\eta \equiv t + \frac{1}{2\omega}\arctan\frac{a+\omega x}{b-\omega y},
\end{aligned}
\tag{33}
$$

where P is a function of $\{\xi,\ \eta\}$ and determined by substituting (33) to (9), i.e.,

$$Q(P) = 4\omega\xi P_{\xi\xi} + 4\omega P_{\xi} + \frac{1}{4\omega\xi} P_{\eta\eta}. \tag{34}$$

In fact, a special case, $a = b = 0,\ \omega = 1$, for (33) and (34) has also been obtained via Darboux transformation in Lou and Li (2006).

To solve (34), one has to fix $Q(P)$ in some particular forms. Here we just take

$$Q(P) = -a_1\omega^2 P. \tag{35}$$

Thanks to the special selection (35), (34) can be resolved by means of the variable separation approach. The result reads

$$P = \sum_{i=1}^{N} \cos\left(2c_i\omega(\eta - \eta_0)\right)\left[d_i J_{c_i}(\rho) + e_i N_{c_i}(\rho)\right], \tag{36}$$

where

$$\rho = \sqrt{a_1(\omega x + a)^2 + a_1(\omega y - b)^2},$$

N is an arbitrary positive integer, $c_i,\ d_i,\ e_i,\ \eta_i,\ (i = 1,\ 2, \ldots, N)$ and a_1 are arbitrary constants, $J_{c_i}(\rho)$ and $N_{c_i}(\rho)$ are c_ith order Bessel and Neumann functions of ρ respectively.

Figure 4a–c display a special Bessel wave structure for the stream, vorticity and velocity respectively with the parameter selections

$$\begin{aligned} &N = a = a_1 = b = 1, \quad \omega = \frac{1}{2}, \quad c_1 = 4, \\ &e_1 = \eta_1 = 0, \quad d_1 = 160 \end{aligned} \tag{37}$$

at time $t = 0$.

In the static case, by taking

$$\Omega = \begin{cases} \lambda^2\psi, & r = \sqrt{x^2 + y^2} \leq R, \\ 0, & r > R, \end{cases} \tag{38}$$

which is different from (35) for $r > R$, the so-called Lamb-dipole solution that is described by a *single* Bessel function solution $J_1(\lambda r)$ inside the circle $r \leq R$ has been obtained (Nielsen and Rasmussen, 1997). Our general solution is different from the Lamb-dipole solution not only in its time dependence, but also in other two aspects, the vorticity is non-zero outside the circle $r > R$ and the solution is described by general superpositions of various arbitrary Bessel waves.

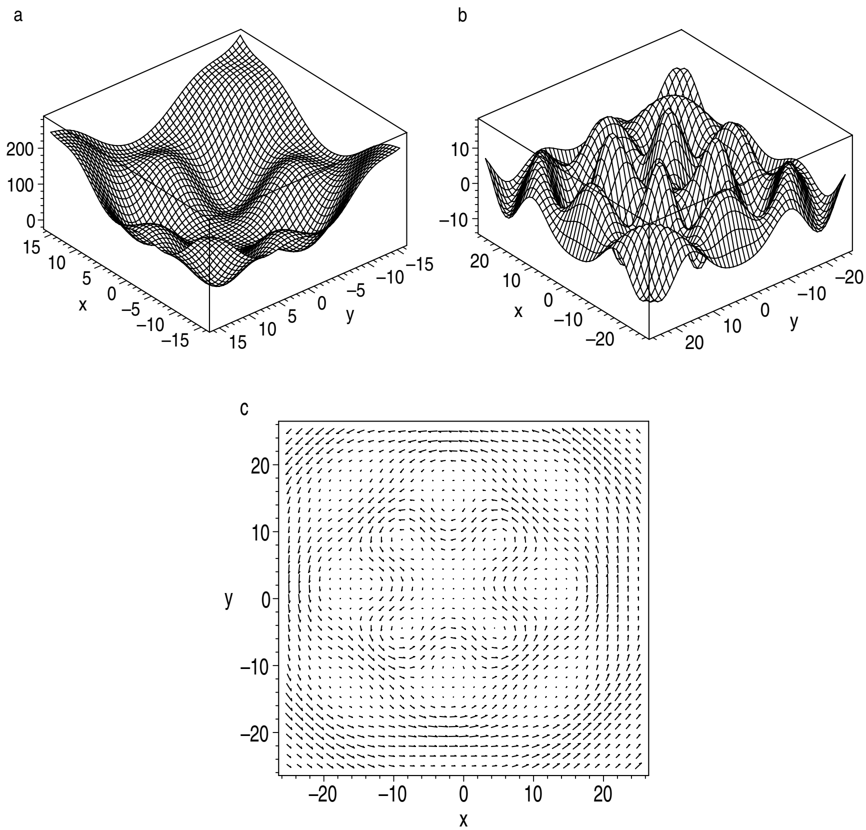

Fig. 4. (a) The structure of a Bessel wave solution for the stream function of the $(2+1)$-dimensional Euler equation expressed by (23) with (36) and (37) at time $t = 0$. (b) and (c) are the plots of vorticity and velocity related to (a).

3. SUMMARY AND DISCUSSIONS

An ordinary Bäcklund transformation theorem is put forward to derive exact explicit analytical solutions though the starting idea is so simple that might be valid for all the equations. The possibility of getting some significant results is decided by a further key step, ansatz (7), which reduces the non-usable and trivial Theorem 1 to a much simpler and applicable Theorem 2. Furthermore, we restrict ourselves to the constant vorticity seed solution both zero and nonzero.

The zero vorticity, equivalent to the space independent velocity field seed solution engenders the interesting solution Theorem 3. From the Theorem 3, we know that any solution of an arbitrary nonlinear Poisson equation will result in an exact solution of the $(2+1)$-dimensional Euler equation, which indicates the ideal fluid (fluid without viscocity) and nearly ideal fluid (fluid with small viscocity) possesses fruitful wave patterns. This is one of the main reasons why various integrable models can be reasonably deduced from the Euler and/or NS equation to approximately describe the real fluid.

Guaranteed by the Theorem 3, making advantage of some special nonlinear Poisson equations such as the Poisson sine equation, some particular types of solitons, solitary waves, periodic waves and two-periodic interaction waves can be created. It is stressed here once again that some special forms of the results presented in this paper have also been produced by the Darboux transformation approach (Lou and Li, 2006) with help of the known weak Lax pair (Li, 2001, 2003; Li and Shvidkoy, 2004; Li and Yurov, 2003).

In the literature (Jia *et al.*, 2006; Lou *et al.*, 2005), two types of the Darboux transformation Theorems have been given for the $(3+1)$-dimensional Euler equation. We believe that similar rich properties of the solution structures for the $(3+1)$-dimensional Euler equation can also be obtained by some types of simple direct methods. Because of the wide applications of the both $(2+1)$- and $(3+1)$-dimensional Euler equations, the more about their exact solutions are worthy of further study.

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